Exact solution of a one-parameter family of asymmetric exclusion processes

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We define a family of asymmetric processes for particles on a one-dimensional lattice, depending on a continuous parameter $\lambda\in[0,1]$, interpolating between the completely asymmetric processes (for $\lambda=1$) and the $n=1$ drop-push models (for $\lambda=0$). For arbitrary λ, the model describes an exclusion process, in which a particle pushes its right neighboring particles to the right, with rates depending on the number of these particles. Using the Bethe ansatz, we obtain the exact solution of the master equation. [S1063-651X(98)12905-7]

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I. INTRODUCTION

Various versions of one-dimensional asymmetric simple exclusion processes (ASEP) have been shown to be of physical interest in a variety of problems including the kinetics of biopolymerization [1], polymers in random media, dynamical models of interface growth [2], and traffic models [3]. This model is also related to the noisy Burgers equation [4], and thus to the study of shocks [5,6]. Besides the equilibrium properties of this model, its dynamical properties have also been studied in [6–8].

Recently the totally ASEP model, with sequencial updating on an infinite lattice, has been solved exactly by Schütz [9] using the coordinate Bethe ansatz. In this model, each lattice site can be occupied by at most one particle and a particle hops with rate one to its right neighboring site if it is not already occupied; otherwise the attempted move is rejected. In his work, instead of using the quantum Hamiltonian formalism, which is suitable for studying the dynamical exponents and certain time-dependent correlation functions, Schütz adopted the coordinate representation for writing the master equation. By solving the master equation exactly, he was able to obtain explicit expressions for conditional probabilities $P(x_1,x_2,\ldots,x_N;t|y_1,y_2,\ldots,y_N;0)$ of finding N particles on lattice sites $x_1,\ldots,x_N$ at time t with initial occupation $y_1,\ldots,y_N$ at time $t=0$.

The master equation for the probability of finding particle 1 on site k_1, particle 2 on site $k_2,\ldots,$ and particle N on site k_N, with $k_N>k_{N-1}>\cdots k_2>k_1$, is written as

$$\begin{aligned}\frac{\partial}{\partial t}P(k_1,k_2,\ldots,k_N,t)&=P(k_1-1,k_2,\ldots,k_N,t)+P(k_1,k_2-1,\ldots,k_N,t)\\&\quad+\cdots+P(k_1,k_2,\ldots,k_N-1,t)\\&\quad-NP(k_1,k_2,\ldots,k_N,t),\end{aligned}\tag{1}$$

if $k_{i+1}-k_i>1$. This equation was then augmented by the following boundary condition:

$$P(k,k,t)=P(k,k+1,t),\quad \forall k.\tag{2}$$

In writing Eq. (2), we have supressed for simplicity the position of all the other particles, bearing in mind that this condition should hold for every pair of adjacent variables k_i and k_{i+1}. In the following we always use this simplified notation. It was then shown that Eqs. (1) and (2) give the correct master equation in the whole physical region (i.e., the region $k_i<k_{i+1}$) for the probabilities. In the rest of [9], the exact solution of the master equation (1) [with boundary condition (2)] is constructed.

We now describe what we have done in the present paper. In Sec. II, we substitute the boundary condition (2) by

$$P(k,k,t)=P(k-1,k,t),\quad \forall k,\tag{3}$$

and show that this boundary condition, together with Eq. (1), describes the $n=1$ drop-push dynamics [10]. In this process, even if the right neighboring sites of a particle are occupied, the particle hops with rate one to the next right site, pushing the right neighboring particles to the next sites. This means that all the following processes occur with equal rate one:

$$\begin{gathered}A0\to 0A,\\ AA0\to 0AA,\\ AAA0\to 0AAA,\\ \vdots\\ \underbrace{AA\cdots A}_{n}0\to 0\underbrace{AA\cdots A}_{n},\end{gathered}\tag{4}$$

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where we have adopted the standard notation for representing a particle by A and a vacancy by 0. We then obtain a closed form for the conditional proabilities for this process.

This process, in which a particle pushes as many particles with rate one, is the opposite extreme of what was solved by Schütz, and interestingly admits a closed form solution for the conditional probabilities $P(x_1,x_2,\ldots,x_N;t|y_1,y_2,\ldots,y_N;0)$ in the form of an $N\times N$ determinant.

In Sec. III, we combine the boundary condition (2) and (3) in the form

$$P(k,k,t)=\lambda P(k,k+1,t)+(1-\lambda)P(k-1,k,t),\quad \forall k, \tag{5}$$

and show that the resulting master equation [(1) and (5)], describes a procession that the processes shown in Eq (4) occur with unequal rates: namely, the process

$$A\underbrace{AA\cdots A}_{n}0\rightarrow 0A\underbrace{AA\cdots A}_{n}, \tag{6}$$

occurs with rate

$$r_n=\frac{1}{1+\lambda/\mu+(\lambda/\mu)^2+\cdots+(\lambda/\mu)^n}, \tag{7}$$

where $\mu=1-\lambda$. We call this model *generalized totally asymmetric exclusion process*. In the limit $\lambda\rightarrow 0$, we have $r_n=1,\forall n$, and in the limit $\lambda=1$, we have $r_0=1$ and $r_{n\neq 0}=0$. Note also that $r_{n+1}\leqslant r_n$, $\forall n$. Therefore this process is perhaps more physical than the two extreme cases studied in [9] and in Sec. II of this paper.

In Sec. IV, we use the coordinate Bethe ansatz and solve the master equation of the process defined in Sec. III, and show that there is no bound state in the spectrum.

In Sec. V, we write the quantum Hamiltonian formalism for the generalized process and, using a particle-hole exchange transformation, show that this generalized process is equivalent (i.e., in the same universality class) to another process, where particles hop only to the left. In this new process, if a left neighboring site is occupied, the move is rejected, but if a set of the left neighboring sites are empty, the particle hops with distance dependent rates to these sites:

$$0A\rightarrow A0 \quad \text{with rate} \quad r_0$$

$$00A\rightarrow A00 \quad \text{with rate} \quad r_1$$

$$\vdots$$

$$0\underbrace{00\cdots 0}_{n}A\rightarrow A0\underbrace{00\cdots 0}_{n} \qquad \text{with rate } r_n. \tag{8}$$

Therefore a transformation as simple as a particle-hole exchange, when applied to our generalized process, has an interesting physical consequence. Models with different values of λ all allow exact solutions in the form of the coordinate Bethe ansatz and their spectra have only the continuous part, but only the limiting cases of these models ($\lambda=0$ and 1) allow a closed solution in the form of a determinant.

We end up the paper with conclusion in Sec. V.

II. GENERALIZED TOTALLY ASYMMETRIC EXCLUSION PROCESS WITH $\lambda=0$

We augment the master equation (1) with the boundary condition (3). Although we derive the rates for arbitrary λ in the next section by a general argument, here we want to show that for $\lambda=0$ case, the master equation (1) [together with the boundary condition (3)] describe an $n=1$ drop-push dynamics. For simplicity, consider the two particle sector of $n=1$ drop push dynamics. The master equations are

$$\frac{\partial}{\partial t}P(k_1,k_2,t)=P(k_1-1,k_2,t)+P(k_1,k_2-1,t)-2P(k_1,k_2,t),\quad k_2>k_1+1, \tag{9}$$

$$\frac{\partial}{\partial t}P(k,k+1,t)=P(k-1,k+1,t)+P(k-1,k,t)-2P(k,k+1,t). \tag{10}$$

Now, if we choose the boundary condition

$$P(k,k,t)=P(k-1,k,t),\quad \forall k, \tag{3$'$}$$

Eq. (10) can be written as

$$\frac{\partial}{\partial t}P(k,k+1,t)=P(k-1,k+1,t)+P(k,k,t)-2P(k,k+1,t), \tag{11}$$

which is of the same form as Eq. (9).

In the three particle sector, the extra equation that needs to be taken into account is

$$\frac{\partial}{\partial t}P(k,k+1,k+2)=P(k-1,k+1,k+2)+P(k-1,k,k+2)+P(k-1,k,k+1)-3P(k,k+1,k+2). \tag{12}$$

Using the boundary condition (3), the second and the third terms on the right-hand side of Eq. (12) can be written as

$$P(k-1,k,k+2)=P(k,k,k+2), \tag{13}$$

$$P(k-1,k,k+1)=P(k,k,k+1)=P(k,k+1,k+1), \tag{14}$$

which means that Eq. (12) is equivalent to the following standard form:

$$\frac{\partial}{\partial t}P(k,k+1,k+2)=P(k-1,k+1,k+2)+P(k,k,k+2)+P(k,k+1,k+1)-3P(k,k+1,k+2). \tag{15}$$

This procedure can be repeated for any sector. We will give a general proof in the next section.

To solve the master equation (1), and the boundary condition (3), for the conditional probability $P(x_1,x_2,\dots,x_N;t|y_1,y_2,\dots,y_N;0)$, we set, following Schütz [9],

$$P(x_1,x_2,\dots,x_N;t|y_1,y_2,\dots,y_N;0)=e^{-Nt}\det G_N, \tag{16}$$

where G_N is an $N\times N$ matrix with entries $G_{ij}=g_{i-j}(x_i-y_j,t)$. The functions $g_p(x,t)$ are to be determined. Writing G_N as

$$G_N=\det\begin{bmatrix} G_1(x_1,t)\\ \vdots\\ G_i(x_i,t)\\ \vdots\\ G_N(x_N,t)\end{bmatrix}, \tag{17}$$

where

$$G_i(x_i)=[g_{i-1}(x_i-y_1,t),g_{i-2}(x_i-y_2,t),\dots,g_{i-N}(x_i-y_N,t)], \tag{18}$$

and inserting Eq. (16) in (1), we obtain

$$\sum_{i=1}^{N}\det\begin{bmatrix} G_1(x_1,t)\\ \vdots\\ \frac{\partial}{\partial t}G_i(x_i,t)\\ \vdots\\ G_N(x_N,t)\end{bmatrix}=\sum_{i=1}^{N}\det\begin{bmatrix} G_1(x_1,t)\\ \vdots\\ G_i(x_i-1,t)\\ \vdots\\ G_N(x_N,t)\end{bmatrix}, \tag{19}$$

the solution of which is

$$\frac{\partial}{\partial t}G_i(x_i,t)=G_i(x_i-1,t). \tag{20}$$

Inserting Eq. (16) in the boundary condition (3), we obtain

$$\det\begin{bmatrix} G_1(x_1,t)\\ \vdots\\ G_{k-1}(x,t)\\ G_k(x,t)\\ \vdots\\ G_N(x_N,t)\end{bmatrix}=\det\begin{bmatrix} G_1(x_1,t)\\ \vdots\\ G_{k-1}(x-1,t)\\ G_k(x,t)\\ \vdots\\ G_N(x_N,t)\end{bmatrix}, \tag{21}$$

the solution of which is

$$G_{k-1}(x,t)=G_{k-1}(x-1,t)+\beta G_k(x,t), \tag{22}$$

where β is an arbitrary parameter. The explicit form of the function $g_p(x,t)$ can now be determined: these functions, as seen by Eqs. (20) and (22), should satisfy the following relations:

$$\frac{\partial}{\partial t}g_p(n,t)=g_p(n-1,t), \tag{23}$$

$$g_p(n,t)=g_p(n-1,t)+\beta g_{p+1}(n,t). \tag{24}$$

Defining the generating functions (or z transforms) $\tilde{g}_p(z,t):=\sum_{n=-\infty}^{\infty}z^n g_p(n,t)$, Eqs. (23) and (24) are converted to

$$\frac{\partial}{\partial t}\tilde{g}_p(z,t)=z\tilde{g}_p(z,t), \tag{25}$$

and

$$\tilde{g}_{p+1}(z,t)=\frac{1}{\beta}(1-z)\tilde{g}_p(z,t), \tag{26}$$

the solution of which is simply obtained as

$$\tilde{g}_p(z,t)=e^{zt}\tilde{g}_p(z,0)=e^{zt}\left(\frac{1-z}{\beta}\right)^p\tilde{g}_0(z,0). \tag{27}$$

$\tilde{g}_0(z,0)$ is nothing but the generating function for $g_0(n,0)$, the one particle sector probabilities at $t=0$. Since $P(x,0|y,0)=g_0(x-y,0)=\delta_{x,y}$, we have $g_0(n,0)=\delta_{n,0}$, and thus $\tilde{g}_0(z,0)=1$, giving finally

$$\tilde{g}_p(z,t)=e^{zt}\left(\frac{1-z}{\beta}\right)^p. \tag{28}$$

The parameter β, as long as it is nonzero, drops out of the determinant and so we can set it equal to unity. The functions $g_p(n,t)$ are obtained by expanding the generating functions. Note that the functions $\tilde{g}_p(z,t)$ should be expanded in terms of positive powers of z, if $p<0$. This is due to the fact that, for $p<0$, as $n\to-\infty$, the function $g_p(n,t)$ tend to zero, since this limit is in the physical region. This expansion yields, formally,

$$g_p(n,t)=\sum_{k=-\infty}^{n}\binom{p}{n-k}\frac{(-1)^{n-k}}{k!}t^k. \tag{29}$$

If $p\geqslant 0$, $g_p(n,t)$ is converted to a finite sum

$$g_{p\geqslant 0}(n,t)=\sum_{k=0}^{\min(n,p)}\binom{p}{k}\frac{(-1)^k}{(n-k)!}t^{n-k}. \tag{30}$$

In particular,

$$g_0(n,t)=\frac{t^n}{n!}. \tag{31}$$

If $p<0$, $g_p(n,t)$ is converted to another finite sum

$$g_p(n,t)=\sum_{k=0}^{n}\binom{|p|+k-1}{|p|-1}\frac{t^{n-k}}{(n-k)!}. \tag{32}$$

We have thus obtained an explicit relation for the conditional probability.

III. GENERALIZED TOTALLY ASYMMETRIC EXCLUSION PROCESS WITH ARBITRARY λ

We now consider the master equation (1) together with the boundary condition

$$P(k,k,t)=\lambda P(k,k+1,t)+\mu P(k-1,k,t), \quad \forall k. \tag{5'}$$

It can be easily shown that the conservation of probability demands that $\mu=1-\lambda$. In order to understand what type of process is described by these equations, we first look at the two particle case. Equations (1) and (5′) yield

$$\begin{aligned}\frac{\partial}{\partial t}P(k,k+1)&=P(k-1,k+1)+P(k,k)-2P(k,k+1)\\&=P(k-1,k+1)+\mu P(k-1,k)\\&\quad-(1+\mu)P(k,k+1),\end{aligned} \tag{33}$$

which means the following rates:

$$A0\to 0A \quad \text{with rate} \quad r_0=1,$$

$$AA0\to 0AA \quad \text{with rate} \quad r_1=\mu.$$

To find the rates in the general case, we first prove a lemma.

Lemma: Equation (5′) implies, for arbitrary n, the following:

$$\begin{aligned}&P(k,k+1,k+2,\dots,k+n-1,k+n,k+n)\\&\quad=(1-r_{n+1})P(k,k+1,k+2,\dots,k+n-1,k\\&\qquad+n,k+n+1)+r_{n+1}P(k-1,k,k+1,\dots,k+n\\&\qquad-2,k+n-1,k+n),\end{aligned} \tag{34}$$

where

$$r_n=\left[1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2+\cdots+\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}. \tag{35}$$

Proof: We proceed by induction. For $n=0$, Eqs. (34) and (35) are the same as Eq. (5′), as $r_1=\mu$. Assuming now that Eqs. (34) and (35) are correct for $n-1$, and using Eq. (5′), we have

$$\begin{aligned}&P(k,k+1,\dots,k+n-1,k+n,k+n)\\&\quad=\lambda P(k,k+1,\dots,k+n-1,k+n,k+n+1)+\mu P(k,k+1,\dots,k+n-1,k+n-1,k+n)\\&\quad=\lambda P(k,k+1,\dots,k+n-1,k+n,k+n+1)+\mu\{(1-r_n)P(k,k+1,\dots,k+n-1,k+n,k+n)\\&\qquad+r_nP(k-1,k,\dots,k+n-2,k+n-1,k+n)\},\end{aligned} \tag{36}$$

or

$$\begin{aligned}&P(k,k+1,\dots,k+n-1,k+n,k+n)\\&\quad=s_{n+1}P(k,k+1,\dots,k+n-1,k+n,k+n+1)\\&\qquad+r_{n+1}P(k-1,k,\dots,k+n-2,k+n-1,k+n),\end{aligned} \tag{37}$$

where

$$\frac{\lambda}{1-\mu(1-r_n)}=s_{n+1}, \quad \frac{\mu r_n}{1-\mu(1-r_n)}=r_{n+1}. \tag{38}$$

From Eq. (38), it is seen that $s_{n+1}+r_{n+1}=1$. One can now solve the second equation of (38) for r_{n+1} to obtain

$$\frac{\mu r_n}{\lambda+\mu r_n}=r_{n+1} \quad \text{or} \quad r_{n+1}^{-1}=\frac{\lambda}{\mu}r_n^{-1}+1,$$

which gives

$$\begin{aligned}r_{n+1}^{-1}&=1+\frac{\lambda}{\mu}\left\{1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2+\cdots+\left(\frac{\lambda}{\mu}\right)^n\right\}\\&=1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2+\cdots+\left(\frac{\lambda}{\mu}\right)^{n+1}.\end{aligned}$$

This proves the lemma.

We now consider a collection of n adjacent particles and write the master equation for this configuration by Eq. (1):

$$\begin{aligned}&\frac{\partial P}{\partial t}(k,k+1,k+2,\dots,k+n-1)\\&\quad=\sum_{i=0}^{n-1}P(k,k+1,\dots,k+i-2,k+i-1,k+i-1,k+i\\&\qquad+1,\dots,k+n-1)-nP(k,k+1,k+2,\dots,k+n-1).\end{aligned} \tag{39}$$

Using Eq. (34), we find

$$\begin{aligned}&\frac{\partial P}{\partial t}(k,k+1,k+2,\dots,k+n-1)\\&\quad=\sum_{i=0}^{n-1}r_iP(k-1,k,\dots,k+i-2,k+i-1,k+i+1,\dots,k\\&\qquad+n-1)-\left(\sum_{i=0}^{n-1}r_i\right)P(k,k+1,k+2,\dots,k+n-1).\end{aligned} \tag{40}$$

It is now obvious that the above equation describes a process in which a collection of $i+1$ adjacent particles hop to the right with rate r_i, as claimed in the Introduction.

IV. THE BETHE ANSATZ SOLUTION FOR ARBITARY λ

In this section we denote the position of the particles by $x_i\in\mathbf{Z}$ rather than k_i, and apply the Bethe ansatz for the solution of the master equation (1) and the boundary condition (5′). Writing $P_N(x_1,\dots,x_N,t)=e^{-\epsilon_N t}\Psi_N(x_1,\dots,x_N)$, will turn Eq. (1) into an eigenvalue equation for $\Psi_N(x_1,\dots,x_N)$:

$$\sum_{i=1}^{N}\Psi_N(x_1,\dots,x_i-1,\dots,x_N)=(N-\epsilon_N)\Psi_N(x_1,\dots,x_i,\dots,x_N). \tag{41}$$

We write the coordinate Bethe ansatz for Ψ in the form:

$$\Psi_N(x_1,\dots,x_N)=\sum_\sigma A_\sigma e^{i\sigma(\mathbf{p})\cdot\mathbf{x}}, \tag{42}$$

where $\mathbf{x}$ and $\mathbf{p}$ stand for the n-tuple coordinates and momenta and $\sigma(\mathbf{p})$ is a permutation of momenta. The sum is over all permutations. Inserting Eq. (42) into Eq. (41) yields

$$\sum_\sigma A_\sigma e^{i\sigma(\mathbf{p})\cdot\mathbf{x}}(e^{-i\sigma(p_1)}+e^{-i\sigma(p_2)}+\cdots+e^{-i\sigma(p_N)})=(N-\epsilon_N)\Psi_N(x_1,\dots,x_N). \tag{43}$$

The sum in the parentheses can be written as $\sum_{k=1}^{N}e^{-ip_k}$ and taken outside Σ_σ, yielding

$$\epsilon_N:=\sum_{k=1}^{N}\epsilon(p_k)=\sum_{k=1}^{N}(1-e^{-ip_k}). \tag{44}$$

Note that due to translational invariance, Ψ_N is also an eigenvector of total momentum P, which in the lattice is defined as the logarithm of the shift operator $U=e^{-iP}$:

$$(U\Psi_N)(x_1,\dots,x_N):=\Psi_N(x_1-1,x_2-1,\dots,x_N-1). \tag{45}$$

Acting by U on Eq. (42), we obtain

$$(P\Psi_N)(x_1,\dots,x_N)=(p_1+\cdots+p_N)\Psi_N(x_1,\dots,x_N). \tag{46}$$

Therefore the eigenvectors Ψ_N have additive total *energies* and momenta. Inserting Eq. (42) in the boundary condition (5′), rewritten in an unabbreviated form

$$\begin{aligned}\Psi(x_1,\dots,x_i=\xi,x_{i+1}=\xi,\dots,x_N)&=\lambda\Psi(x_1,\dots,x_i=\xi,x_{i+1}=\xi+1,\dots,x_N)\\&\quad+\mu\Psi(x_1,\dots,x_i=\xi-1,x_{i+1}=\xi,\dots,x_N),\end{aligned}$$

we obtain

$$\sum_\sigma e^{i\Sigma_{k\neq i,i+1}\sigma(p_k)x_k+i[\sigma(p_i)+\sigma(p_{i+1})]\xi}[A_\sigma(1-\lambda e^{i\sigma(p_{i+1})}-\mu e^{-i\sigma(p_i)})]=0. \tag{47}$$

We denote the expression in the bracket by B_σ. Noting that the prefactor is unaffected by an interchange of p_i and p_{i+1}, it follows that the proper coefficient of each prefactor, which should vanish, is $B_\sigma+B_{\sigma\sigma_i}$, where σ_i is the generator of S_N (the permutation group of N object), which only interchanges p_i and p_{i+1}

$$\sigma_i(p_1,\dots,p_i,p_{i+1},\dots,p_N)=(p_1,\dots,p_{i+1},p_i,\cdots,p_N), \tag{48}$$

and $\sigma\sigma_i$ stands for the product of two group elements, σ acting after σ_i. Therefore we find

$$A_\sigma(1-\lambda e^{i\sigma(p_{i+1})}-\mu e^{-i\sigma(p_i)})+A_{\sigma\sigma_i}(1-\lambda e^{i\sigma(p_i)}-\mu e^{-i\sigma(p_{i+1})})=0,$$

or

$$\frac{A_{\sigma\sigma_i}}{A_\sigma}=\frac{\lambda e^{i\sigma(p_{i+1})}+\mu e^{-i\sigma(p_i)}-1}{1-\lambda e^{i\sigma(p_i)}-\mu e^{-i\sigma(p_{i+1})}}=S[\sigma(p_i),\sigma(p_{i+1})]. \tag{49}$$

This relation, in effect, allows one to find all the A_σ's in terms of A_1 (which is set to unity). The first few coefficients, corresponding to the elements $1,\sigma_1,\sigma_2,\sigma_1\sigma_2,\sigma_2\sigma_1,\sigma_1\sigma_2\sigma_1$ are

$$\begin{gathered}A_1=1,\quad A_{\sigma_1}=S_{12},\quad A_{\sigma_2}=S_{23},\\A_{\sigma_1\sigma_2}=S_{12}S_{13},\quad A_{\sigma_2\sigma_1}=S_{13}S_{23},\quad A_{\sigma_1\sigma_2\sigma_1}=S_{12}S_{13}S_{23},\end{gathered} \tag{50}$$

where $S_{ij}=S(p_i,p_j)$. The form of the scattering matrix S_{ij} could also be found from the two particle sector alone. The above analysis shows in fact the factorizibility of the S matrix in the general case, a sign of the integrability of the problem.

To find the range of p_i's, we analyze the S matrix,

$$S_{12}=\frac{\lambda e^{ip_2}+\mu e^{-ip_1}-1}{1-\lambda e^{ip_1}-\mu e^{-ip_2}}=\frac{c_{21}}{c_{12}}, \tag{51}$$

and the two particle wave function

$$\Psi_2(x_1,x_2)=c_{12}e^{i(p_1x_1+p_2x_2)}+c_{21}e^{i(p_2x_1+p_1x_2)},$$

or

$$\Psi(X,x)=e^{iPX}(c_{12}e^{ipx}+c_{21}e^{-ipx}), \tag{52}$$

where $X:=\frac{1}{2}(x_1+x_2)$, $x:=x_1-x_2$, $P:=p_1+p_2$, and $p:=\frac{1}{2}(p_1-p_2)$ with clear physical meanings. Since x is negative ($x_1<x_2$), to have a bound state one of the following set of conditions should be satisfied simultaneously. Either

$$c_{12}=0,\quad \operatorname{Im}p>0,\quad \operatorname{Im}P=0,$$

or

$$c_{21}=0,\quad \operatorname{Im}p<0,\quad \operatorname{Im}P=0.$$

Rewriting c_{12} in terms of the new momenta, we find

$$c_{12}=e^{ip}(e^{-ip}-\lambda e^{iP/2}-\mu e^{-iP/2}). \tag{53}$$

Since $\mathrm{Im}P=0$ we have

$$|\lambda e^{iP/2}+\mu e^{-iP/2}|\leqslant\lambda+\mu=1. \tag{54}$$

Noting that $|e^{-ip}|>1$, it is seen that c_{12} cannot vanish. A similar analysis applies for the second set of conditions. Therefore no bound state exists in the spectrum, and the range of all momentum variables is $[0,2\pi)$.

To find the conditional probability $P_N(x_1,x_2,\ldots,x_N;t|y_1,y_2,\ldots y_N;0)$, one should take a linear combination of the eigenfunctions Ψ_N, with suitable coefficients. Consider the two particle sector. We have

$$P_2(x_1,x_2;t|y_1,y_2;0)=\int_0^{2\pi}\int_0^{2\pi}\frac{dp_1}{2\pi}\frac{dp_2}{2\pi}\times e^{-[\epsilon(p_1)+\epsilon(p_2)]t-ip_1y_1-ip_2y_2}\times\Psi_2(x_1,x_2). \tag{55}$$

This is just a linear combination of the eigenfunctions, satisfying the initial condition

$$P_2(x_1,x_2;0|y_1,y_2;0)=\delta_{x_1,y_1}\delta_{x_2,y_2},$$

in the physical region $(x_2>x_1,y_2>y_1)$. The eigenfunction $\Psi_2(x_1,x_2)$ in Eq. (55) is normalized according to

$$\Psi_2(x_1,x_2)=e^{i(p_1x_1+p_2x_2)}+S_{12}e^{i(p_2x_1+p_1x_2)}.$$

To avoid the singularity in S_{12}, we set $p_1\to p_1+i\epsilon$. With this prescription, the contribution of the second term in Ψ_2 to $P_2(x_1,x_2,0|y_1,y_2,0)$ identically vanishes in the physical region. Using the variables $\xi:=e^{ip_1}$ and $\eta:=e^{-ip_2}$, a simple contour integration yields

$$P_2(x_1,x_2;t|y_1,y_2;0)=e^{-2t}\left\{\frac{t^{x_1-y_1}}{(x_1-y_1)!}\frac{t^{x_2-y_2}}{(x_2-y_2)!}-\sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m}\lambda^m\mu^{k-m}\frac{t^{x_2-y_1+m}}{(x_2-y_1+m)!}\frac{t^{x_1-y_2-k+m}}{(x_1-y_2-k+m)!}\times\left[1-\frac{\lambda t}{x_1-y_2-k+m+1}-\frac{\mu(x_2-y_1+m)}{t}\right]\right\}. \tag{56}$$

It is easy to see, explicitly, that this solution satisfies the initial condition in the physical region. Also, in the limiting cases $\lambda=1$ and $\lambda=0$, it reduces, respectively, to

$$P_2(x_1,x_2;t|y_1,y_2;0)=e^{-2t}\left\{\frac{t^{x_1-y_1}}{(x_1-y_1)!}\frac{t^{x_2-y_2}}{(x_2-y_2)!}-\left[\frac{t^{x_1-y_2}}{(x_1-y_2)!}-\frac{t^{x_1-y_2+1}}{(x_1-y_2+1)!}\right]\sum_{k=0}^{\infty}\frac{t^{x_2-y_1+k}}{(x_2-y_1+k)!}\right\}, \tag{57}$$

obtained in [9], and

$$P_2(x_1,x_2;t|y_1;y_2,0)=e^{-2t}\left\{\frac{t^{x_1-y_1}}{(x_1-y_1)!}\frac{t^{x_2-y_2}}{(x_2-y_2)!}-\left[\frac{t^{x_2-y_1}}{(x_2-y_1)!}-\frac{t^{x_2-y_1-1}}{(x_2-y_1-1)!}\right]\sum_{k=0}^{\infty}\frac{t^{x_1-y_2-k}}{(x_1-y_2-k)!}\right\}, \tag{58}$$

obtained in the present paper.

The treatment of the N particle case is similar. We have

$$P_N(x_1,\ldots,x_N;t|y_1,\ldots,y_N;0)=\int_0^{2\pi}\frac{dp_1}{2\pi}\cdots\int_0^{2\pi}\frac{dp_N}{2\pi}e^{-[\Sigma\epsilon(p_i)]t-i\Sigma p_iy_i}\Psi_N(x_1,\ldots,x_N).$$

The integration is defined with the following ϵ prescription: in S_{ij} $(i<j)$, p_i is replaced by $p_i+i\epsilon$.

V. HAMILTONIAN APPROACH

The Hilbert space of generalized totally asymmetric exclusion process is $\mathcal{H}=\otimes C_2$, the tensor product of all the local Hilbert spaces of the lattice sites. C_2 is the two dimensional vector space with basis states $|0\rangle=\binom{1}{0}$ and $|1\rangle=\binom{0}{1}$. The states $|0\rangle$ and $|1\rangle$ represent vaccant and occupied sites, respectively. The local operators $n_i=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, $\sigma_i^+=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, and $\sigma_i^-=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ are the number, annihilation, and creation operators, respectively. Their action on a bra state $\langle\alpha|$, $(\alpha=0,1)$, can be conveniently represented as $\langle\alpha|n=\alpha\langle\alpha|$, $\langle\alpha|\sigma^+=(1-\alpha)\langle1-\alpha|$ and $\langle\alpha|\sigma^-=\alpha\langle1-\alpha|$.

The state of the system $|\Psi(t)\rangle$ evolves according to the Schrödinger type equation $-\partial/\partial t|\Psi(t)\rangle=H|\Psi(t)\rangle$. The connection between the two representations is given by the relation

$$P(k_1,k_2,\ldots,k_N,t)=\langle k_1,k_2,\ldots,k_N|\Psi_N(t)\rangle=\langle0|\sigma_{k_1}^+\sigma_{k_2}^+,\ldots,\sigma_{k_N}^+|\Psi_N(t)\rangle. \tag{59}$$

The Hamiltonian of the process can now be written as

$$H=-\sum_{k\in L}\sum_{l\geq 1} r_{l-1}[v_k(l)-w_k(l)], \tag{60}$$

where L represents the sites of the lattice and

$$v_k(l)=\sigma_k^+ n_{k+1}n_{k+2}\cdots n_{k+l-1}\sigma_{k+l}^-, \tag{61}$$

$$w_k(l)=n_k n_{k+1}n_{k+2}\cdots n_{k+l-1}(1-n_{k+l}). \tag{62}$$

Consider a bra state containing l particles on adjacent sites: $\langle k+1,k+2,\ldots,k+l|$. The only terms in H with nonvanishing action on this state are

$$\begin{aligned}\langle k+1,k+2,\ldots,k+l|v_k(i)&\\ =\langle k,k+1,\ldots,k+i-1,k+i+1,\ldots,k+l|,&\\ 1\leq i\leq l,&\end{aligned} \tag{63}$$

$$\begin{aligned}\langle k+1,k+2,\ldots,k+l|w_{k+1+l-i}(i)&\\ =\langle k+1,k+2,\ldots,k+l|,\quad 1\leq i\leq l.&\end{aligned} \tag{64}$$

Note that the action of the above operators on every other state that contains, beside the above particles, other collection of particles disconnected from the above one is the same. Using Eqs. (59)–(64), one arrives at Eq. (40) for the evolution of the probability. Note that the quantum Hamiltonian (60) is a stochastic operator, meaning that all of its off-diagonal matrix elements are nonpositive with the sum of entries in each column being equal to zero. This last property is expressed by saying that $\langle S|H=0$ where $\langle S|$ is the sum of all basis states of $\mathcal{H}$. Equivalent models may be obtained by constructing operators $\Omega:\mathcal{H}\rightarrow\mathcal{H}$ and Hamiltonians $H'=\Omega H\Omega^{-1}$, which preserve the above properties. An obvious example is the particle-hole exchange operator $\Omega=\Pi_i\sigma_i^x$. It clearly has the property that $\langle S|\Omega=\langle S|$, so that for H' we also have $\langle S|H'=0$. It is easy to see that this transformation induces the changes $n\leftrightarrow 1-n$ and $\sigma^+\leftrightarrow\sigma^-$. So the master equation obtained from H' describes the process (8).

VI. DISCUSSION AND OUTLOOK

We have defined a generalized exclusion process, parametrized by a real parameter $\lambda\in[0,1]$, and have shown that the master equation of this model admits for every λ an exact solution via the coordinate Bethe ansatz. We have also shown that this model interpolates continuously between two very different models: the totally asymmetric exclusion model (for $\lambda=1$), which we may consider as the weak coupling limit and the drop-push model (for $\lambda=0$), which may be considered as the strong coupling limit of the model. In these two limits, the solution acquires a simple determinant form.

Our work can be further investigated in one definite way. It may be that the point $\lambda=\frac{1}{2}$ is a point of phase transition and the study of the equilibrium properties of the model on a periodic lattice may reveal this transition. There are already two pieces of evidence for the validity of this conjecture. First, there is some sort of duality between two models with parameters symmetric with respect to $\lambda=1/2$. To be more specific, we have

$$S(p_1,p_2;\lambda)=S(-p_2,-p_1;1-\lambda). \tag{65}$$

Second, the large l behavior of the transition rates is

$$r_l\sim\begin{cases}1-\dfrac{\lambda}{\mu}, & \lambda<\dfrac{1}{2}\\[2mm] \dfrac{1}{l}, & \lambda=\dfrac{1}{2}\\[2mm] \left(\dfrac{\lambda}{\mu}\right)^{-l}, & \lambda>\dfrac{1}{2}.\end{cases}$$

It will be interesting to study the stationary behavior of this system along the lines that have been followed in [11–14], to see what kind of phases develop in the system by varying λ.

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